Rupert Counterexample

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Chapter 1

Introduction

We follow for the most part the structure of [SY25].

Chapter 2

The Noperthedron

2.1 Definition of the Noperthedron

We define three points $C_1, C_2, C_3 \in \mathbb{Q}^3$.

$$C_1 := \frac{1}{259375205}\begin{pmatrix}152024884\\0\\210152163\end{pmatrix}, \qquad C_2 := \frac{1}{10^{10}}\begin{pmatrix}6632738028\\6106948881\\3980949609\end{pmatrix},$$

$$C_3 := \frac{1}{10^{10}}\begin{pmatrix}8193990033\\5298215096\\1230614493\end{pmatrix}.$$

Lemma 1. $\|C_1\| = 1$, $\frac{98}{100} < \|C_2\| < \frac{99}{100}$, *and* $\frac{98}{100} < \|C_3\| < \frac{99}{100}$.

Proof. Trivial arithmetic. $\square$

Lemma 2. *The radius of the Noperthedron is one.*

Proof. By Theorem 1, Theorem 24, **??**, and Theorem 7. $\square$

Rotations about the x, y, z axes $R_x, R_y, R_z : \mathbb{R} \to \mathbb{R}^{3\times 3}$ are defined in the usual way:

$$R_x(\alpha) := \begin{pmatrix}1 & 0 & 0\\0 & \cos\alpha & -\sin\alpha\\0 & \sin\alpha & \cos\alpha\end{pmatrix}, \qquad R_y(\alpha) := \begin{pmatrix}\cos\alpha & 0 & -\sin\alpha\\0 & 1 & 0\\\sin\alpha & 0 & \cos\alpha\end{pmatrix},$$

$$R_z(\alpha) := \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\\sin\alpha & \cos\alpha & 0\\0 & 0 & 1\end{pmatrix}.$$

Where Steininger and Yurkevich define a 30-element set C_30

$$\mathcal{C}_{30} := \left\{(-1)^\ell R_z\left(\frac{2\pi k}{15}\right) : k = 0, \dots, 14; \ell = 0, 1\right\}.$$

of rotations, we instead define

Definition 3.

$$\mathcal{C}_{15} := \left\{ R_z\left(\frac{2\pi k}{15}\right) : k = 0, \dots, 14 \right\}.$$

without point-symmetricness 'baked in' as it is in C_{30}. It's more convenient for the formalization to apply C_{15} to the points C_1, C_2, C_3, and then point-symmetrize that set afterwards.

Definition 4. A set $S \subseteq \mathbb{R}^3$ is *point-symmetric* if $x \in S$ implies $-x \in S$.

Definition 5. The *pointsymmetrization* of a collection of vertices $v_1, \dots, v_n \in \mathbb{R}^3$ is $v_1, \dots, v_n, -v_1, \dots, -v_n$.

We write $\mathcal{C}_{15} \cdot P = \{cP \text{ for } c \in \mathcal{C}_{15}\}$ for the orbit of P under the action of $\mathcal{C}_{15}$.

Definition 6. The Noperthedron is polyhedron given by the vertex set that is the pointsymmetrization of

$$\mathcal{C}_{15} \cdot C_1 \cup \mathcal{C}_{15} \cdot C_2 \cup \mathcal{C}_{15} \cdot C_3$$

Lemma 7. *The norm of any vertex in the prepointsymmetrized version of the Noperthedron is no more than 1.*

Proof. Evident from definitions. □

Lemma 8. *The pointsymmetrization of any set is point-symmetric.*

Proof. Evident from definitions. □

Lemma 9. *The noperthedron is point-symmetric.*

Proof. Follows from Lemma 8. □

2.2 Refined Rupert's property for the Noperthedron

Lemma 10. *Let* $\mathbf{P} = \mathbf{NOP}$*, then for all* $\theta, \varphi, \alpha \in \mathbb{R}$*, the following three identities hold (as sets):*

$$\begin{aligned} M(\theta + 2\pi/15, \varphi) \cdot \mathbf{P} &= M(\theta, \varphi) \cdot \mathbf{P}, \\ R(\alpha + \pi) M(\theta, \varphi) \cdot \mathbf{P} &= R(\alpha) M(\theta, \varphi) \cdot \mathbf{P}, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M(\theta, \varphi) \cdot \mathbf{P} &= M(\theta + \pi/15, \pi - \varphi) \cdot \mathbf{P}\,. \end{aligned}$$

Proof. See [SY25], Lemma 7. □

Corollary 11. *If the noperthedron is Rupert, then there exists a solution with*

$$\begin{aligned} \theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58]. \end{aligned}$$

Proof. See [SY25], Lemma 8. □

Chapter 3

Bounding Rotations

Lemma 12. *For any* $\alpha, \theta, \varphi \in \mathbb{R}$ *and* $a \in \{x, y, z\}$ *one has* $\|R(\alpha)\| = \|R_a(\alpha)\| = \|M(\theta, \varphi)\| = 1$.

Proof. See [SY25], Lemma 9. □

Lemma 13. *Let* $\varepsilon > 0$, $|\alpha - \overline{\alpha}| \leq \varepsilon$ *and* $a \in \{x, y, z\}$ *then* $\|R_a(\alpha) - R_a(\overline{\alpha})\| = \|R(\alpha) - R(\overline{\alpha})\| < \varepsilon$.

Proof. See [SY25], Lemma 10. □

Lemma 14. *For all* $a, b \in \mathbb{R}$ *with* $|a|, |b| \leq 2$ *the following inequality holds:*

$$(1 + \cos(a))(1 + \cos(b)) \geq 2 + 2\cos\left(\sqrt{a^2 + b^2}\right),$$

with equality only for $a = 0$ *or* $b = 0$.

Proof. Use the Jensen inequality. See [SY25], Lemma 11. □

Lemma 15. *For any* $\alpha, \beta \in \mathbb{R}$ *one has*

$$\|R_x(\alpha)R_y(\beta) - \mathrm{Id}\,\| \leq \sqrt{\alpha^2 + \beta^2}$$

with equality only for $\alpha = \beta = 0$.

Proof. See [SY25], Lemma 12. □

Lemma 16. *Let* $\varepsilon > 0$ *and* $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$ *then* $\|M(\theta, \varphi) - M(\overline{\theta}, \overline{\varphi})\|, \|X(\theta, \varphi) - X(\overline{\theta}, \overline{\varphi})\| < \sqrt{2}\varepsilon$.

Proof. See [SY25], Lemma 13. □

Lemma 17. *Let* $P \in \mathbb{R}^3$ *with* $\|P\| \leq 1$. *Further, let* $\varepsilon > 0$ *and* $\overline{\theta}, \overline{\varphi}, \theta, \varphi \in \mathbb{R}$ *such that* $|\overline{\theta} - \theta|, |\overline{\varphi} - \varphi| \leq \varepsilon$. *If* $\langle X(\overline{\theta}, \overline{\varphi}), P\rangle > \sqrt{2}\varepsilon$ *then* $\langle X(\theta, \varphi), P\rangle > 0$.

Proof. See [SY25], Lemma 14. □

Lemma 18. *Let* $P \in \mathbb{R}^3$ *with* $\|P\| \leq 1$. *Further, let* $\varepsilon, r > 0$ *and* $\overline{\theta}, \overline{\varphi}, \theta, \varphi \in \mathbb{R}$ *such that* $|\overline{\theta} - \theta|, |\overline{\varphi} - \varphi| \leq \varepsilon$. *If* $\|M(\overline{\theta}, \overline{\varphi})P\| > r + \sqrt{2}\varepsilon$ *then* $\|M(\theta, \varphi)P\| > r$.

Proof. See [SY25], Lemma 15. □

Lemma 19. *Let* $\varepsilon > 0$ *and* $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}|, |\alpha - \overline{\alpha}| \leq \varepsilon$ *then* $\|R(\alpha)M(\theta, \varphi) - R(\overline{\alpha})M(\overline{\theta}, \overline{\varphi})\| < \sqrt{5}\varepsilon$.

Proof. See [SY25], Lemma 16. □

Chapter 4

Preliminaries

TODO: This whole chapter needs organization, it's just a grab bag of miscellaneous results for now.

4.1 Rupert Sets

Theorem 20 (Rupert Polyhedron iff Rupert Set)**.** *The following are equivalent:*

- *The convex polyhedron with vertex set v is Rupert.*
- *The convex closure of v is a Rupert set.*

Proof. TODO: import this from the other repo ☐

4.2 Poses

TODO

Theorem 21. *Given a pose with zero offset, there exists a view pose that is equivalent to it.*

4.3 Pointsymmetry and Rupertness

Theorem 22. *If a set is point symmetric and convex, then it being Rupert implies it being purely rotationally Rupert.*

Proof. TODO: informalize proof ☐

Theorem 23. *Suppose S is a finite set of points in $\mathbb{R}^n$. The radius of the polyhedron S is r iff*

- *there is a vector $v \in S$ with $\|v\| = r$*
- *all vectors $v \in S$ have $\|v\| \leq r$*

Proof. Immediate from definition. ☐

Theorem 24. *Pointsymmetrization preserves radius.*

Proof. Because the reflection of a point about the origin preserves its norm. ☐

Chapter 5

The Global Theorem

Lemma 25. *Suppose $V = V_1, \dots, V_m \subseteq \mathbb{R}^n$ be a finite sequence of points. Suppose $\mathsf{Co}(V)$ is its convex hull. Let $S \in \mathsf{Co}(V)$ and $w \in \mathbb{R}^n$ be given. then*

$$\langle S, w\rangle \leq \max_i \langle V_i, w\rangle$$

Proof. This is a mild generalization of [SY25], Lemma 18.

Since $S \in \mathsf{Co}(V)$, we have

$$S = \sum_{j=1}^m \lambda_j V_j$$

for some $\lambda_1, \dots, \lambda_m \in [0, 1]$ with

$$1 = \sum_{j=1}^m \lambda_j$$

Therefore

$$\langle S, w\rangle = \left\langle \sum_{j=1}^m \lambda_j V_j, w \right\rangle = \sum_{j=1}^m \lambda_j \left\langle V_j, w\right\rangle \leq \sum_{j=1}^m \lambda_j \max_i \langle V_i, w\rangle$$

$$= \max_i \langle V_i, w\rangle \sum_{j=1}^m \lambda_j = \max_i \langle V_i, w\rangle$$

as required. $\square$

Lemma 26. *Let $S \in \mathbb{R}^3$ and $w \in \mathbb{R}^2$ be unit vectors and set $f(x_1, x_2, x_3) = \langle R(x_3) M(x_1, x_2) S, w\rangle$. Then for all $x_1, x_2, x_3 \in \mathbb{R}$ and any $i, j \in \{1, 2, 3\}$ it holds that*

$$\left| \frac{\mathrm{d}^2 f}{\mathrm{d}x_i \mathrm{d}x_j}(x_1, x_2, x_3) \right| \leq 1.$$

Proof. See [SY25], Lemma 19. $\square$

Lemma 27. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ such that $|x_1 - y_1|, \dots, |x_n - y_n| \leq \varepsilon$. If $\left|\partial_{x_i} \partial_{x_j} f(x_1, \dots, x_n)\right| \leq 1$ for all $i, j \in \{1, \dots, n\}$ then*

$$|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)| \leq \varepsilon \sum_{i=1}^n |\partial_{x_i} f(x_1, \dots, x_n)| + \frac{n^2}{2}\varepsilon^2.$$

Proof. See [SY25], Lemma 20. □

Theorem 28 (Global Theorem)**.** *Let* $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* $\rho = 1$ *and let* $S \in \mathbf{P}$*. Further let* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$ *and let* $w \in \mathbb{R}^2$ *be a unit vector. Denote* $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^{\theta} := M^{\theta}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_1}^{\varphi} := M^{\varphi}(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^{\theta}, \overline{M_2}^{\varphi}$*. Finally set*

$$G := \langle R(\overline{\alpha})\overline{M_1}S, w\rangle - \varepsilon \cdot (|\langle R'(\overline{\alpha})\overline{M_1}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\theta}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\varphi}S, w\rangle|) - 9\varepsilon^2/2,$$

$$H_P := \langle \overline{M_2}P, w\rangle + \varepsilon \cdot (|\langle \overline{M_2}^{\theta}P, w\rangle| + |\langle \overline{M_2}^{\varphi}P, w\rangle|) + 2\varepsilon^2, \quad \text{for } P \in \mathbf{P}\,.$$

If $G > \max_{P \in \mathbf{P}} H_P$ *then there does not exist a solution to Rupert's condition with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in U := [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5\,.$$

Proof. See [SY25], Section 4.2. □

Chapter 6

The Local Theorem

Lemma 29. *For any* $P \in \mathbb{R}^3$ *one has* $\left\|M(\theta,\varphi)P\right\|^2 = \|P\|^2 - \langle X(\theta,\varphi), P\rangle^2$.

Proof. See [SY25], Lemma 21. $\square$

Definition 30. Given $v_1, \dots, v_n \in \mathbb{R}^n$ write $\operatorname{span}^+(v_1, \dots, v_n)$ for the set (simplicial cone) in $\mathbb{R}^n$ defined by

$$\operatorname{span}^+(v_1, \dots, v_n) = \left\{ w \in \mathbb{R}^n : \exists \lambda_1, \dots, \lambda_n > 0 \text{ s.t. } w = \sum_{i=1}^{n} \lambda_i v_i \right\},$$

which is the natural restriction of $\operatorname{span}(v_1, \dots, v_n)$ to positive weights.

Lemma 31. *Let* $V_1, V_2, V_3, Y, Z \in \mathbb{R}^3$ *with* $\|Y\| = \|Z\|$ *and* $Y, Z \in \operatorname{span}^+(V_1, V_2, V_3)$. *Then at least one of the following inequalities does not hold:*

$$\begin{aligned} \langle V_1, Y\rangle &> \langle V_1, Z\rangle, \\ \langle V_2, Y\rangle &> \langle V_2, Z\rangle, \\ \langle V_3, Y\rangle &> \langle V_3, Z\rangle. \end{aligned}$$

Proof. See [SY25], Lemma 23. $\square$

Lemma 32. *For* $A, \overline{A}, B, \overline{B} \in \mathbb{R}^{n\times n}$ *and* $P_1, P_2 \in \mathbb{R}^n$ *it holds that*

$$|\langle AP_1, BP_2\rangle - \langle \overline{A}P_1, \overline{B}P_2\rangle| \le \|P_1\| \cdot \|P_2\| \cdot \left(\|A - \overline{A}\| \cdot \|\overline{B}\| + \|\overline{A}\| \cdot \|B - \overline{B}\| + \|A - \overline{A}\| \cdot \|B - \overline{B}\| \right).$$

Proof. See [SY25], Lemma 24. $\square$

Lemma 33. *For* $A, B \in \mathbb{R}^{n\times n}$ *and* $P_1, P_2 \in \mathbb{R}^n$ *one has*

$$|\langle AP_1, AP_2\rangle - \langle BP_1, BP_2\rangle| \le \|P_1\| \cdot \|P_2\| \cdot \|A - B\| \cdot \left(\|A\| + \|B\| + \|A - B\| \right).$$

Proof. See [SY25], Lemma 25. $\square$

Lemma 34. *Let* $A, B, C \in \mathbb{R}^2$ *be such that* $\langle R(\pi/2)A, B\rangle, \langle R(\pi/2)B, C\rangle, \langle R(\pi/2)C, A\rangle > 0$. *Then the origin lies strictly in the triangle* ABC.

Proof. See [SY25], Lemma 26. $\square$

Definition 35. Let $\theta, \varphi \in \mathbb{R}$, $\varepsilon > 0$, and set $M := M(\theta, \varphi)$. Three points $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ are called ε-spanning for (θ, φ) if it holds that:

$$\begin{aligned}\langle R(\pi/2)MP_1, MP_2\rangle &> 2\varepsilon(\sqrt{2}+\varepsilon),\\ \langle R(\pi/2)MP_2, MP_3\rangle &> 2\varepsilon(\sqrt{2}+\varepsilon),\\ \langle R(\pi/2)MP_3, MP_1\rangle &> 2\varepsilon(\sqrt{2}+\varepsilon).\end{aligned}$$

Lemma 36. *Let $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ be ε-spanning for $(\overline{\theta}, \overline{\varphi})$ and let $\theta, \varphi \in \mathbb{R}$ such that $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$. Assume that $\langle X(\theta, \varphi), P_i\rangle > 0$ for $i = 1, 2, 3$. Then*

$$X(\theta, \varphi) \in \operatorname{span}^+(P_1, P_2, P_3).$$

Proof. See [SY25], Lemma 28. □

Lemma 37. *Let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$. Let $\varepsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$, then set*

$$T := \left(R(\overline{\alpha})M(\overline{\theta}_1, \overline{\varphi}_1)P + M(\overline{\theta}_2, \overline{\varphi}_2)Q\right)/2 \in \mathbb{R}^2,$$

and $\delta \geq \|T - M(\overline{\theta}_2, \overline{\varphi}_2)Q\|$. Finally, let $\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha \in \mathbb{R}$ with $|\overline{\theta}_1 - \theta_1|, |\overline{\varphi}_1 - \varphi_1|, |\overline{\theta}_2 - \theta_2|, |\overline{\varphi}_2 - \varphi_2|, |\overline{\alpha} - \alpha| \leq \varepsilon$. Then $R(\alpha)M(\theta_1, \varphi_1)P, M(\theta_2, \varphi_2)Q \in \operatorname{Disc}_{\delta+\sqrt{5}\varepsilon}(T)$.

Proof. See [SY25], Lemma 30. □

Definition 38. Let $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon and $Q \in \mathcal{P}$ one of its vertices. Assume that for some $\overline{Q} \in \mathbb{R}^2$ it holds that $Q \in \operatorname{Disc}_\delta(\overline{Q})$, i.e. $\|Q - \overline{Q}\| < \delta$. Define $\operatorname{Sect}_\delta(\overline{Q}) := \operatorname{Disc}_\delta(\overline{Q}) \cap \mathcal{P}^\circ$ as the intersection between $\operatorname{Disc}_\delta(\overline{Q})$ and the interior of the convex hull of $\mathcal{P}$.

Moreover, $Q \in \mathcal{P}$ is called *δ-locally maximally distant with respect to $\overline{Q}$ (δ-LMD($\overline{Q}$))* if for all $A \in \operatorname{Sect}_\delta(\overline{Q})$ it holds that $\|Q\| > \|A\|$.

Lemma 39. *Let $\mathcal{P}$ be a convex polygon and $Q \in \mathcal{P}$ be one of its vertices. Let $\overline{Q} \in \mathbb{R}^2$ with $\|Q - \overline{Q}\| < \delta$ for some $\delta > 0$. Assume that for some $r > 0$ such that $\|Q\| > r$ it holds that*

$$\frac{\langle Q, Q - P_j\rangle}{\|Q\|\|Q - P_j\|} \geq \delta/r,$$

for all other vertices $P_j \in \mathcal{P} \setminus Q$. Then $Q \in \mathcal{P}$ is δ-locally maximally distant with respect to $\overline{Q}$.

Proof. See [SY25], Lemma 32. □

Lemma 40. *Let $\varepsilon > 0$ and $\theta, \overline{\theta}, \varphi, \overline{\varphi} \in \mathbb{R}$ with $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$. Define $M = M(\theta, \varphi)$ and $\overline{M} = M(\overline{\theta}, \overline{\varphi})$ and let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$. Then:*

$$\frac{\langle MP, M(P-Q)\rangle}{\|MP\| \cdot \|M(P-Q)\|} \geq \frac{\langle \overline{M}P, \overline{M}(P-Q)\rangle - 2\varepsilon\|P-Q\| \cdot (\sqrt{2}+\varepsilon)}{(\|\overline{M}P\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M}(P-Q)\| + 2\sqrt{2}\varepsilon)}.$$

Proof. See [SY25], Lemma 33. □

Theorem 41 (Local Theorem)**.** *Let $\mathbf{P}$ be a polyhedron with radius $\rho = 1$ and $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$ be not necessarily distinct. Assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent.*

Let $\varepsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$, then set $\overline{X_1} := X(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} := X(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that

$$(-1)^{\sigma_P}\langle \overline{X_1}, P_i\rangle > \sqrt{2}\varepsilon \quad \text{and} \quad (-1)^{\sigma_Q}\langle \overline{X_2}, Q_i\rangle > \sqrt{2}\varepsilon, \tag{A$_\varepsilon$}$$

for all $i = 1, 2, 3$. *Moreover, assume that* P_1, P_2, P_3 *are* ε*-spanning for* $(\overline{\theta}_1, \overline{\varphi}_1)$ *and that* Q_1, Q_2, Q_3 *are* ε*-spanning for* $(\overline{\theta}_2, \overline{\varphi}_2)$. *Finally, assume that for all* $i = 1, 2, 3$ *and any* $Q_j \in \mathbf{P} \setminus Q_i$ *it holds that*

$$\frac{\langle \overline{M_2} Q_i, \overline{M_2}(Q_i - Q_j)\rangle - 2\varepsilon \|Q_i - Q_j\| \cdot (\sqrt{2} + \varepsilon)}{(\|\overline{M_2} Q_i\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M_2}(Q_i - Q_j)\| + 2\sqrt{2}\varepsilon)} > \frac{\sqrt{5}\varepsilon + \delta}{r}, \tag{B$_\varepsilon$}$$

for some $r > 0$ *such that* $\min_{i=1,2,3} \|\overline{M_2} Q_i\| > r + \sqrt{2}\varepsilon$ *and for some* $\delta \in \mathbb{R}$ *with*

$$\delta \geq \max_{i=1,2,3} \left\|R(\overline{\alpha})\overline{M_1} P_i - \overline{M_2} Q_i\right\| / 2.$$

Then there exists no solution to Rupert's problem $R(\alpha) M(\theta_1, \varphi_1)\,\mathbf{P} \subset M(\theta_2, \varphi_2)\,\mathbf{P}^{\circ}$ *with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \coloneqq U \subseteq \mathbb{R}^5\,.$$

Proof. See [SY25], Theorem 36. □

Chapter 7

Rational Versions

Definition 42. We define the two functions $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ by:

$$\begin{aligned}\sin_{\mathbb{Q}}(x) &:= x - \frac{x^3}{3} + \frac{x^5}{5!} \mp \cdots + \frac{x^{25}}{25!},\\ \cos_{\mathbb{Q}}(x) &:= 1 - \frac{x^2}{2} + \frac{x^4}{4!} \mp \cdots + \frac{x^{24}}{24!}.\end{aligned}$$

Further, by replacing $\sin, \cos$ with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ we define the functions

$$R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta,\varphi), M_{\mathbb{Q}}(\theta,\varphi), M^{\theta}_{\mathbb{Q}}(\theta,\varphi), M^{\varphi}_{\mathbb{Q}}(\theta,\varphi).$$

Lemma 43.

$$|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \frac{|x|^{27}}{27!} \quad \textit{and} \quad |\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \frac{|x|^{26}}{26!}.$$

Proof. Appeal to Taylor series bounds, using the fact that all absolute values of higher derivatives of sine and cosine never exceed 1. □

Lemma 44. *For every $x \in [-4, 4]$ it holds that*

$$|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \frac{\kappa}{7} \quad \textit{and} \quad |\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \frac{\kappa}{7}.$$

Proof. Straightforward numerical calculation from Lemma 43. □

Lemma 45. *Let $A = (a_{i,j})_{1\leq i\leq m,\ 1\leq j\leq n} \in \mathbb{R}^{m\times n}$ and $\delta > 0$. Assume that $|a_{i,j}| \leq \delta$. Then it holds that $\|A\| \leq \delta\sqrt{mn}$.*

Proof. For any $v \in \mathbb{R}^n$ we have

$$\|Av\|^2 = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i,j}v_j\right)^2 \leq \sum_{i=1}^{m}\left(\sum_{j=1}^{n} \delta|v_j|\right)^2 = \delta^2 m \left(\sum_{j=1}^{n}|v_j|\right)^2 \leq \delta^2 mn\|v\|^2$$

using the Cauchy-Schwarz inequality. Dividing by $\|v\|$ and taking the square root proves the claim. □

Lemma 46. *Let $A(x,y)$ be an $m \times n$ matrix with $1 \le m,n \le 3$ such that every entry is in $[-1,1]$.of the form $a_1(x) \cdot a_2(y)$ where $a_i(z) \in [-1,1]$. Define $A_{\mathbb{Q}}(x,y)$ by replacing* sin *with* $\sin_{\mathbb{Q}}$ *and* cos *with* $\cos_{\mathbb{Q}}$. *Then for every $x, y \in [-4,4]$ it holds that $\|A(x,y) - A_{\mathbb{Q}}(x,y)\| \le \kappa$.*

Proof. We've replaced the assumption $a_i(z) \in \{0, 1, -1, \pm\sin(z), \pm\cos(z)\}$ in [SY25]'s Lemma 40 with $a_i(z) \in [-1,1]$.

By assumption, for fixed x, y every entry of $A(x,y) - A_{\mathbb{Q}}(x,y)$ is of the form $ab - \tilde{a}\tilde{b}$ for some $a, b \in [-1,1]$ and $|a - \tilde{a}|, |b - \tilde{b}| \le \kappa/7$ by Theorem 44. This implies that

$$|ab - \tilde{a}\tilde{b}| \le |ab - a\tilde{b}| + |a\tilde{b} - \tilde{a}\tilde{b}| = |a| \cdot |b - \tilde{b}| + |\tilde{b}| \cdot |a - \tilde{a}| \le 1 \cdot \kappa/7 + (1 + \kappa/7) \cdot \kappa/7 < \kappa/3.$$

So we can apply Theorem 45 and obtain that $\|A(x,y) - A_{\mathbb{Q}}(x,y)\| < \kappa/3 \cdot \sqrt{3 \cdot 3} = \kappa$. □

Corollary 47. *Let $\alpha, \theta, \varphi \in [-4,4]$. Then it holds that*

$$\begin{aligned} &\|R(\alpha) - R_{\mathbb{Q}}(\alpha)\|, \|R'(\alpha) - R'_{\mathbb{Q}}(\alpha)\|, \|X(\theta,\varphi) - X_{\mathbb{Q}}(\theta,\varphi)\|, \|M(\theta,\varphi) - M_{\mathbb{Q}}(\theta,\varphi)\|, \\ &\qquad \|M^{\theta}(\theta,\varphi) - M^{\theta}_{\mathbb{Q}}(\theta,\varphi)\|, \|M^{\varphi}(\theta,\varphi) - M^{\varphi}_{\mathbb{Q}}(\theta,\varphi)\| \le \kappa. \end{aligned}$$

Moreover,

$$\|R_{\mathbb{Q}}(\alpha)\|, \|R'_{\mathbb{Q}}(\alpha)\|, \|X_{\mathbb{Q}}(\theta,\varphi)\|, \|M_{\mathbb{Q}}(\theta,\varphi)\|, \|M^{\theta}_{\mathbb{Q}}(\theta,\varphi)\|, \|M^{\varphi}_{\mathbb{Q}}(\theta,\varphi)\| \le 1 + \kappa$$

Proof. The first statement is a direct application of Theorem 46 and the second statement follows immediately after using Theorem 12 and the triangle inequality. □

Lemma 48. *For $1 \le i \le n$ let (A_i, B_i) be pairs of real matrices, such that for each i the dimensions of A_i and B_i are equal. Assume moreover that the products $A_1 \cdots A_n$ and $B_1 \cdots B_n$ are well defined. Finally, assume that $\|A_i - B_i\| \le \kappa$ and let $\delta_i \ge \max(\|A_i\|, \|B_i\|, 1)$. Then it holds that $\|A_1 \cdots A_n - B_1 \cdots B_n\| \le n\kappa \cdot \delta_1 \cdots \delta_n$.*

Proof. See [SY25], Lemma 42. □

Lemma 49. *Let $\alpha, \theta, \varphi \in [-4,4]$, $P \in \mathbb{R}^3$ with $\|P\| \le 1$ and let $\widetilde{P}$ be a κ-rational approximation of P. Set $M = M(\theta,\varphi)$ and $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta,\varphi)$, $M^{\theta} = M^{\theta}(\theta,\varphi)$, $M^{\theta}_{\mathbb{Q}} = M^{\theta}_{\mathbb{Q}}(\theta,\varphi)$, $M^{\varphi} = M^{\varphi}(\theta,\varphi)$, $M^{\varphi}_{\mathbb{Q}} = M^{\varphi}_{\mathbb{Q}}(\theta,\varphi)$ as well as $R = R(\alpha)$, $R_{\mathbb{Q}} = R_{\mathbb{Q}}(\alpha)$, $R' = R'(\alpha)$, $R'_{\mathbb{Q}} = R'_{\mathbb{Q}}(\alpha)$. Finally let $w \in \mathbb{R}^2$ with $\|w\| = 1$. Then:*

$$|\langle MP, w\rangle - \langle M_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 3\kappa, \tag{7.1}$$

$$|\langle M^{\theta}P, w\rangle - \langle M^{\theta}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 3\kappa, \tag{7.2}$$

$$|\langle M^{\varphi}P, w\rangle - \langle M^{\varphi}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 3\kappa, \tag{7.3}$$

$$|\langle RMP, w\rangle - \langle R_{\mathbb{Q}}M_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa, \tag{7.4}$$

$$|\langle R'MP, w\rangle - \langle R'_{\mathbb{Q}}M_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa, \tag{7.5}$$

$$|\langle RM^{\theta}P, w\rangle - \langle R_{\mathbb{Q}}M^{\theta}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa, \tag{7.6}$$

$$|\langle RM^{\varphi}P, w\rangle - \langle R_{\mathbb{Q}}M^{\varphi}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa. \tag{7.7}$$

Proof. See [SY25], Lemma 44. □

Theorem 50 (Rational Global Theorem)**.** *Let* $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* $\rho = 1$ *and* $\widetilde{\mathbf{P}}$ *a* κ*-rational approximation. Let* $\widetilde{S} \in \widetilde{\mathbf{P}}$*. Further let* $\varepsilon > 0$ *and* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$*. Let* $w \in \mathbb{Q}^2$ *be a unit vector. Denote* $\overline{M_1} := M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_2} := M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^{\theta} := M_{\mathbb{Q}}^{\theta}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_1}^{\varphi} := M_{\mathbb{Q}}^{\varphi}(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^{\theta}, \overline{M_2}^{\varphi}$*. Finally set*

$$\begin{aligned} G^{\mathbb{Q}} &:= \langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle - \varepsilon \cdot \left(|\langle R'_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^{\theta}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^{\varphi}\widetilde{S}, w\rangle|\right) \\ &\qquad - 9\varepsilon^2/2 - 4\kappa(1 + 3\varepsilon), \\ H_P^{\mathbb{Q}} &:= \langle \overline{M_2}P, w\rangle + \varepsilon \cdot \left(|\langle \overline{M_2}^{\theta}P, w\rangle| + |\langle \overline{M_2}^{\varphi}P, w\rangle|\right) + 2\varepsilon^2 + 3\kappa(1 + 2\varepsilon). \end{aligned}$$

If $G^{\mathbb{Q}} > \max_{P \in \widetilde{\mathbf{P}}} H_P^{\mathbb{Q}}$ *then there does not exist a solution to Rupert's condition to* $\mathbf{P}$ *with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon].$$

Proof. □

Definition 51. Let $\theta, \varphi \in \mathbb{Q} \cap [-4, 4]$ and $M_{\mathbb{Q}} := M_{\mathbb{Q}}(\theta, \varphi)$. Three points $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3 \in \mathbb{Q}^3$ with $\|\widetilde{P}_1\|, \|\widetilde{P}_2\|, \|\widetilde{P}_3\| \leq 1 + \kappa$ are called *ε-κ-spanning for* (θ, φ) if it holds that:

$$\begin{aligned} \langle R(\pi/2)M_{\mathbb{Q}}\widetilde{P}_1, M_{\mathbb{Q}}\widetilde{P}_2\rangle &> 2\varepsilon(\sqrt{2} + \varepsilon) + 6\kappa, \\ \langle R(\pi/2)M_{\mathbb{Q}}\widetilde{P}_2, M_{\mathbb{Q}}\widetilde{P}_3\rangle &> 2\varepsilon(\sqrt{2} + \varepsilon) + 6\kappa, \\ \langle R(\pi/2)M_{\mathbb{Q}}\widetilde{P}_3, M_{\mathbb{Q}}\widetilde{P}_1\rangle &> 2\varepsilon(\sqrt{2} + \varepsilon) + 6\kappa. \end{aligned}$$

Lemma 52. *Let* $P_1, P_2, P_3 \in \mathbb{R}^3$ *with* $\|P_i\| \leq 1$ *and* $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3 \in \mathbb{Q}^3$ *be their* κ*-rational approximations. Assume that* $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ *are* ε*-*κ*-spanning for some* $\theta, \varphi \in \mathbb{Q} \cap [-4, 4]$*, then* P_1, P_2, P_3 *are* ε*-spanning for* θ, φ*.*

Proof. See [SY25], Lemma 46. □

Lemma 53. *Let* $P, Q \in \mathbb{R}^3$ *with* $\|P\|, \|Q\| \leq 1$ *and* $\widetilde{P}, \widetilde{Q}$ *some respective* κ*-rational approximations. Moreover, let* $\alpha, \theta, \varphi \in \mathbb{R} \in [-4, 4]$ *and set* $X = X(\theta, \varphi)$*,* $X_{\mathbb{Q}} = X_{\mathbb{Q}}(\theta, \varphi)$ *as well as* $M = M(\theta, \varphi)$*,* $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta, \varphi)$*. Then*

$$|\langle X, P\rangle - \langle X_{\mathbb{Q}}, \widetilde{P}\rangle| \leq 3\kappa, \tag{7.8}$$

$$|\langle MP, MQ\rangle - \langle M_{\mathbb{Q}}\widetilde{P}, M_{\mathbb{Q}}\widetilde{Q}\rangle| \leq 5\kappa, \tag{7.9}$$

$$|\|MQ\| - \|M_{\mathbb{Q}}\widetilde{Q}\|| \leq 3\kappa. \tag{7.10}$$

Proof. See [SY25], Lemma 49. □

Corollary 54. *In the setting of Theorem 53 let additionally* $\overline{\theta}, \overline{\varphi} \in \mathbb{R} \cap [-4, 4]$ *and set* $\overline{M} = M(\overline{\theta}, \overline{\varphi})$*,* $\overline{M}_{\mathbb{Q}} = M_{\mathbb{Q}}(\overline{\theta}, \overline{\varphi})$*. Then*

$$|\|R(\alpha)MP - \overline{M}Q\| - \|R_{\mathbb{Q}}(\alpha)M_{\mathbb{Q}}\widetilde{P} - \overline{M}_{\mathbb{Q}}\widetilde{Q}\|| \leq 6\kappa.$$

Proof. See [SY25], Corollary 50. □

Corollary 55. *In the setting of Theorem 53, let $\sqrt[+]{x}$ be an upper-$\mathbb{Q}$-square-root function and set $\|x\|_+ := \sqrt[+]{\|x\|^2}$. Set*

$$A = \frac{\langle MP, M(P-Q)\rangle - 2\varepsilon\|P-Q\| \cdot (\sqrt{2}+\varepsilon)}{(\|MP\| + \sqrt{2}\varepsilon) \cdot (\|M(P-Q)\| + 2\sqrt{2}\varepsilon)}$$

as well as

$$A_{\mathbb{Q}} = \frac{\langle M_{\mathbb{Q}}\widetilde{P}, M_{\mathbb{Q}}(\widetilde{P}-\widetilde{Q})\rangle - 10\kappa - 2\varepsilon(\|\widetilde{P}-\widetilde{Q}\|_+ + 2\kappa) \cdot (\sqrt{2}+\varepsilon)}{(\|M_{\mathbb{Q}}\widetilde{P}\|_+ + \sqrt{2}\varepsilon + 3\kappa) \cdot (\|M_{\mathbb{Q}}(\widetilde{P}-\widetilde{Q})\|_+ + 2\sqrt{2}\varepsilon + 6\kappa)}.$$

Then it holds that $A \geq A_{\mathbb{Q}}$.

Proof. See [SY25], Corollary 51. □

Theorem 56 (Rational Local Theorem)**.** *Let $\mathbf{P}$ be a polyhedron with radius $\rho = 1$ and $\widetilde{P}_i$ be a κ-rational approximation of $P_i \in \mathbf{P}$. Set $\widetilde{\mathbf{P}} = \{\widetilde{P}_i \text{ for } P_i \in \mathbf{P}\}$. Let $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$ be not necessarily distinct and assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent. Let $\varepsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$. Set $\overline{X_1} := X_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} := X_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M_1} := M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} := M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that*

$$(-1)^{\sigma_P}\langle \overline{X_1}, \widetilde{P}_i\rangle > \sqrt{2}\varepsilon + 3\kappa \quad \text{and} \quad (-1)^{\sigma_Q}\langle \overline{X_2}, \widetilde{Q}_i\rangle > \sqrt{2}\varepsilon + 3\kappa, \tag{$\mathrm{A}_\varepsilon^{\mathbb{Q}}$}$$

for all $i = 1, 2, 3$. Moreover, assume that $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ are ε-κ-spanning for $(\overline{\theta}_1, \overline{\varphi}_1)$ and that $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3$ are ε-κ-spanning for $(\overline{\theta}_2, \overline{\varphi}_2)$. Let $\sqrt[+]{x}$ and $\sqrt[-]{x}$ be upper- and lower-$\mathbb{Q}$-square-root functions, then set $\|Z\|_+ := \sqrt[+]{\|Z\|^2}$ and $\|Z\|_- := \sqrt[-]{\|Z\|^2}$ for $Z \in \mathbb{Q}^n$. Finally, assume that for all $i = 1, 2, 3$ and any $\widetilde{Q}_j \in \widetilde{\mathbf{P}} \setminus \widetilde{Q}_i$ it holds that

$$\frac{\langle \overline{M_2}\widetilde{Q}_i, \overline{M_2}(\widetilde{Q}_i - \widetilde{Q}_j)\rangle - 10\kappa - 2\varepsilon(\|\widetilde{Q}_i - \widetilde{Q}_j\|_+ + 2\kappa) \cdot (\sqrt{2}+\varepsilon)}{(\|\overline{M_2}\widetilde{Q}_i\|_+ + \sqrt{2}\varepsilon + 3\kappa) \cdot (\|\overline{M_2}(\widetilde{Q}_i - \widetilde{Q}_j)\|_+ + 2\sqrt{2}\varepsilon + 6\kappa)} > \frac{\sqrt{5}\varepsilon + \delta}{r}, \tag{$\mathrm{B}_\varepsilon^{\mathbb{Q}}$}$$

for some $r > 0$ such that $\min_{i=1,2,3} \|\overline{M_2}\widetilde{Q}_i\|_- > r + \sqrt{2}\varepsilon + 3\kappa$ and for some $\delta \in \mathbb{R}$ with

$$\delta = \max_{i=1,2,3} \left\| R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{P}_i - \overline{M_2}\widetilde{Q}_i \right\|_+ / 2 + 3\kappa.$$

Then there exists no solution to Rupert's problem $R(\alpha)M(\theta_1, \varphi_1)\,\mathbf{P} \subset M(\theta_2, \varphi_2)\,\mathbf{P}^\circ$ with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5.$$

Proof. , □

Chapter 8

Computational Step

Theorem 57. *There exists a valid solution table with some row that covers*

$$\begin{aligned}\theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42],\\ \varphi_1 &\in [0, \pi] \subset [0, 3.15],\\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58],\\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58].\end{aligned}$$

Proof. By exhibiting the table and running the validity checking algorithm. □

Theorem 58. *For any valid row in a valid solution table, there can be no Rupert solution in the pose interval of that row.*

Proof. Either appeal recursively to this same theorem if the row splits into other nodes in the tree, or appeal to the rational global theorem (Theorem 50) or the rational local theorem (Theorem 56) at the leaves. □

Chapter 9

Main Theorems

Theorem 59. *There does not in fact exist a noperthedron Rupert solution with*

$$\begin{aligned}
\theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\
\varphi_1 &\in [0, \pi] \subset [0, 3.15], \\
\varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\
\alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58].
\end{aligned}$$

Proof. By 57, there is a valid solution table containing a valid row whose pose interval is a superset of the 5-d interval above. By 58, this means there is no Rupert solution in that interval. □

Theorem 60. *There is no view pose that makes the noperthedron have the Rupert property.*

Proof. Theorem 59 says there is no tight view pose that makes the noperthedron Rupert. Corollary 11 says that this suffices for the general case. □

Theorem 61. *There is no purely rotational pose that makes the noperthedron have the Rupert property.*

Proof. Suppose there were a purely rotational pose. Then convert that to an equivalent view pose with Theorem 21 and appeal to Theorem 60. □

Theorem 62. *There is no pose that makes the noperthedron have the Rupert property.*

Proof. By Theorem 22, we need only show that the noperthedron is pointsymmetric to see that if it is Rupert, then it must be Rupert via a purely rotational pose. But Lemma 9 shows exactly this. And yet we know via Theorem 61 that the noperthedron is not rotationally Rupert, so we have a contradiction, hence the noperthedron has no pose that makes it Rupert. □

Theorem 63. *The noperthedron is not a Rupert set.*

Proof. By Theorem 62, there is no pose that makes the noperthedron a Rupert set. □

Theorem 64. *The noperthedron is not a Rupert polyhedron.*

Proof. By Theorem 20 it suffices to show that the convex hull of the noperthedron vertices is not a Rupert set. But this is exactly what Theorem 63 shows. □

Bibliography

[SY25] Jakob Steininger and Sergey Yurkevich. A convex polyhedron without rupert's property. https://arxiv.org/abs/2508.18475, 2025.