Rupert Counterexample

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Chapter 1

Introduction

We follow for the most part the structure of [SY25].

Chapter 2

The Noperthedron

2.1 Definition of the Noperthedron

We define three points $C_1, C_2, C_3 \in \mathbb{Q}^3$.

$$C_1 := \frac{1}{259375205} \begin{pmatrix} 152024884 \\ 0 \\ 210152163 \end{pmatrix}, \qquad C_2 := \frac{1}{10^{10}} \begin{pmatrix} 6632738028 \\ 6106948881 \\ 3980949609 \end{pmatrix},$$

$$C_3 := \frac{1}{10^{10}} \begin{pmatrix} 8193990033 \\ 5298215096 \\ 1230614493 \end{pmatrix}.$$

Lemma 1. $\|C_1\| = 1$, $\frac{98}{100} < \|C_2\| < \frac{99}{100}$, *and* $\frac{98}{100} < \|C_3\| < \frac{99}{100}$.

Proof. Trivial arithmetic. □

Lemma 2. *The radius of the Noperthedron is one.*

Proof. By Theorem 1, **??**, and **??**. □

Rotations about the x, y, z axes $R_x, R_y, R_z : \mathbb{R} \to \mathbb{R}^{3\times 3}$ are defined in the usual way:

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \qquad R_y(\alpha) := \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix},$$

$$R_z(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We define a 30-element set C_{30}

$$\mathcal{C}_{30} := \left\{ (-1)^\ell R_z\left(\frac{2\pi k}{15}\right) : k = 0, \dots, 14; \ell = 0, 1 \right\}.$$

of rotations.

We write $\mathcal{C}_{30} \cdot P = \{cP \text{ for } c \in \mathcal{C}_{30}\}$ for the orbit of P under the action of $\mathcal{C}_{30}$.

Definition 3. The Noperthedron is polyhedron given by the vertex set

$$\mathcal{C}_{30} \cdot C_1 \cup \mathcal{C}_{30} \cdot C_2 \cup \mathcal{C}_{30} \cdot C_3$$

Lemma 4. *The norm of any vertex in the Noperthedron is no more than 1.*

Proof. Evident from definitions. □

Definition 5. A set $S \subseteq \mathbb{R}^3$ is *point-symmetric* if $x \in S$ implies $-x \in S$.

Lemma 6. *The noperthedron is point-symmetric.*

Proof. Follows from Lemma **??**. □

2.2 Refined Rupert's property for the Noperthedron

Lemma 7. *Let* $\mathbf{P} = \mathbf{NOP}$*, then for all* $\theta, \varphi, \alpha \in \mathbb{R}$*, the following three identities hold (as sets):*

$$\begin{aligned} M(\theta + 2\pi/15, \varphi) \cdot \mathbf{P} &= M(\theta, \varphi) \cdot \mathbf{P}, \\ R(\alpha + \pi) M(\theta, \varphi) \cdot \mathbf{P} &= R(\alpha) M(\theta, \varphi) \cdot \mathbf{P}, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M(\theta, \varphi) \cdot \mathbf{P} &= M(\theta + \pi/15, \pi - \varphi) \cdot \mathbf{P}\,. \end{aligned}$$

Proof. See [SY25], Lemma 7. □

Corollary 8. *If the noperthedron is Rupert, then there exists a solution with*

$$\begin{aligned} \theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58]. \end{aligned}$$

Proof. See [SY25], Lemma 8. □

Chapter 3

Bounding Rotations

Lemma 9. *For any $\alpha, \theta, \varphi \in \mathbb{R}$ and $a \in \{x, y, z\}$ one has $\|R(\alpha)\| = \|R_a(\alpha)\| = \|R'(\alpha)\| = \|M(\theta, \varphi)\| = 1$ and $\|M^\theta(\theta, \varphi)\|, \|M^\varphi(\theta, \varphi)\| \leq 1$.*

Proof. See [SY25], Lemma 9. □

Lemma 10. *Let $\varepsilon > 0$, $|\alpha - \overline{\alpha}| \leq \varepsilon$ and $a \in \{x, y, z\}$ then $\|R_a(\alpha) - R_a(\overline{\alpha})\| = \|R(\alpha) - R(\overline{\alpha})\| < \varepsilon$.*

Proof. See [SY25], Lemma 10. □

Lemma 11. *For all $a, b \in \mathbb{R}$ with $|a|, |b| \leq 2$ the following inequality holds:*

$$(1 + \cos(a))(1 + \cos(b)) \geq 2 + 2\cos\left(\sqrt{a^2 + b^2}\right),$$

with equality only for $a = 0$ or $b = 0$.

Proof. Use the Jensen inequality. See [SY25], Lemma 11. □

Lemma 12. *For any $|\alpha|, |\beta| \leq 2$ and any distinct coordinate axes $d, d' \in \{x, y, z\}$ one has*

$$\|R_d(\alpha)R_{d'}(\beta) - \operatorname{Id}\| \leq \sqrt{\alpha^2 + \beta^2}$$

with equality only for $\alpha = \beta = 0$.

Proof. See [SY25], Lemma 12. □

Lemma 13. *For any $\alpha, \beta \in \mathbb{R}$ one has*

$$\|R_x(\alpha)R_y(\beta) - \operatorname{Id}\| \leq \sqrt{\alpha^2 + \beta^2}$$

with equality only for $\alpha = \beta = 0$.

Proof. See [SY25], Lemma 12. □

Lemma 14. *Let $\varepsilon > 0$ and $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$ then $\|M(\theta, \varphi) - M(\overline{\theta}, \overline{\varphi})\|, \|X(\theta, \varphi) - X(\overline{\theta}, \overline{\varphi})\| < \sqrt{2}\varepsilon$.*

Proof. See [SY25], Lemma 13. □

Lemma 15. *Let $P \in \mathbb{R}^3$ with $\|P\| \leq 1$. Further, let $\varepsilon > 0$ and $\overline{\theta}, \overline{\varphi}, \theta, \varphi \in \mathbb{R}$ such that $|\overline{\theta} - \theta|, |\overline{\varphi} - \varphi| \leq \varepsilon$. If $\langle X(\overline{\theta}, \overline{\varphi}), P\rangle > \sqrt{2}\varepsilon$ then $\langle X(\theta, \varphi), P\rangle > 0$.*

Proof. See [SY25], Lemma 14. □

Lemma 16. *Let $P \in \mathbb{R}^3$ with $\|P\| \leq 1$. Further, let $\varepsilon, r > 0$ and $\overline{\theta}, \overline{\varphi}, \theta, \varphi \in \mathbb{R}$ such that $|\overline{\theta} - \theta|, |\overline{\varphi} - \varphi| \leq \varepsilon$. If $\|M(\overline{\theta}, \overline{\varphi})P\| > r + \sqrt{2}\varepsilon$ then $\|M(\theta, \varphi)P\| > r$.*

Proof. See [SY25], Lemma 15. Corrigendum: the triangle inquality only implies greater than *or equal to*. □

Lemma 17. *Let $\varepsilon > 0$ and $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}|, |\alpha - \overline{\alpha}| \leq \varepsilon$ then $\|R(\alpha)M(\theta, \varphi) - R(\overline{\alpha})M(\overline{\theta}, \overline{\varphi})\| < \sqrt{5}\varepsilon$.*

Proof. See [SY25], Lemma 16. □

Chapter 4

Preliminaries

TODO: This whole chapter needs organization, it's just a grab bag of miscellaneous results for now.

4.1 Rupert Sets

Theorem 18 (Rupert Polyhedron iff Rupert Set)**.** *The following are equivalent:*

- *The convex polyhedron with vertex set v is Rupert.*
- *The convex closure of v is a Rupert set.*

Proof. TODO: import this from the other repo □

4.2 Poses

TODO

Theorem 19. *Given a pose with zero offset, there exists a 5-parameter pose that is equivalent to it.*

Proof. By putting the pose into a canonical form as a Z rotation followed by a Y followed by a Z. □

4.3 Pointsymmetry and Rupertness

Theorem 20. *If a set is point symmetric and convex, then it being Rupert implies it being purely rotationally Rupert.*

Proof. TODO: informalize proof □

Theorem 21. *Suppose S is a finite set of points in $\mathbb{R}^n$. The radius of the polyhedron S is r iff*

- *there is a vector $v \in S$ with $\|v\| = r$*
- *all vectors $v \in S$ have $\|v\| \leq r$*

Proof. Immediate from definition. □

Chapter 5

The Global Theorem

Lemma 22. *Suppose $V = V_1, \dots, V_m \subseteq \mathbb{R}^n$ be a finite sequence of points. Suppose $\mathsf{Co}(V)$ is its convex hull. Let $S \in \mathsf{Co}(V)$ and $w \in \mathbb{R}^n$ be given. then*

$$\langle S, w\rangle \leq \max_i \langle V_i, w\rangle$$

Proof. This is a mild generalization of [SY25], Lemma 18.

Since $S \in \mathsf{Co}(V)$, we have

$$S = \sum_{j=1}^{m} \lambda_j V_j$$

for some $\lambda_1, \dots, \lambda_m \in [0,1]$ with

$$1 = \sum_{j=1}^{m} \lambda_j$$

Therefore

$$\langle S, w\rangle = \left\langle \sum_{j=1}^{m} \lambda_j V_j, w \right\rangle = \sum_{j=1}^{m} \lambda_j \left\langle V_j, w\right\rangle \leq \sum_{j=1}^{m} \lambda_j \max_i \langle V_i, w\rangle$$

$$= \max_i \langle V_i, w\rangle \sum_{j=1}^{m} \lambda_j = \max_i \langle V_i, w\rangle$$

as required. □

Lemma 23. *Let $S \in \mathbb{R}^3$ and $w \in \mathbb{R}^2$ be unit vectors and set $f(x_1, x_2, x_3) = \langle R(x_3) M(x_1, x_2) S, w\rangle$. Then for all $x_1, x_2, x_3 \in \mathbb{R}$ and any $i, j \in \{1, 2, 3\}$ it holds that*

$$\left| \frac{\mathrm{d}^2 f}{\mathrm{d}x_i \mathrm{d}x_j}(x_1, x_2, x_3) \right| \leq 1.$$

Proof. See [SY25], Lemma 19. □

Lemma 24. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ such that $|x_1 - y_1|, \dots, |x_n - y_n| \leq \varepsilon$. If $\left|\partial_{x_i} \partial_{x_j} f(v)\right| \leq 1$ for all $i, j \in \{1, \dots, n\}$ and all $v \in \mathbb{R}^n$, then*

$$|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)| \leq \varepsilon \sum_{i=1}^{n} |\partial_{x_i} f(x_1, \dots, x_n)| + \frac{n^2}{2}\varepsilon^2.$$

Proof. See [SY25], Lemma 20. □

Lemma 25. *The partial derivatives of all relevant rotations, projections, and inner products used in the Global Theorem are as expected. Specifically:*

- $$f^{\alpha}(\theta,\varphi,\alpha) = \langle R'(\alpha)M(\theta,\varphi)S, w\rangle$$
- $$f^{\theta}(\theta,\varphi,\alpha) = \langle R(\alpha)M^{\theta}(\theta,\varphi)S, w\rangle$$
- $$f^{\varphi}(\theta,\varphi,\alpha) = \langle R(\alpha)M^{\varphi}(\theta,\varphi)S, w\rangle$$
- $$g^{\theta}(\theta,\varphi) = \langle M^{\theta}(\theta,\varphi)P, w\rangle$$
- $$g^{\varphi}(\theta,\varphi) = \langle M^{\varphi}(\theta,\varphi)P, w\rangle$$

where $f(\theta,\varphi,\alpha) = \langle R(\alpha)M(\theta,\varphi)S/\|S\|, w\rangle$ *and* $g(\theta,\varphi) = \langle M(\theta,\varphi)P/\|P\|, w\rangle$.

Proof. By basic properties of derivatives. □

Theorem 26 (Global Theorem)**.** *Let* $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* $\rho = 1$ *and let* $S \in \mathbf{P}$. *Further let* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$ *and let* $w \in \mathbb{R}^2$ *be a unit vector. Denote* $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^{\theta} := M^{\theta}(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_1}^{\varphi} := M^{\varphi}(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^{\theta}, \overline{M_2}^{\varphi}$. *Finally set*

$$G := \langle R(\overline{\alpha})\overline{M_1}S, w\rangle - \varepsilon \cdot \left(|\langle R'(\overline{\alpha})\overline{M_1}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\theta}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\varphi}S, w\rangle|\right) - 9\varepsilon^2/2,$$
$$H_P := \langle \overline{M_2}P, w\rangle + \varepsilon \cdot \left(|\langle \overline{M_2}^{\theta}P, w\rangle| + |\langle \overline{M_2}^{\varphi}P, w\rangle|\right) + 2\varepsilon^2, \qquad \text{for } P \in \mathbf{P}\,.$$

If $G > \max_{P\in\mathbf{P}} H_P$ *then there does not exist a solution to Rupert's condition with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in U := [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5\,.$$

Proof. See [SY25], Section 4.2. □

Chapter 6

The Local Theorem

Lemma 27. *For any* $P \in \mathbb{R}^3$ *one has* $\left\|M(\theta,\varphi)P\right\|^2 = \|P\|^2 - \langle X(\theta,\varphi), P\rangle^2$.

Proof. See [SY25], Lemma 21. $\square$

Definition 28. Given $v_1, \dots, v_n \in \mathbb{R}^n$ write $\operatorname{span}^+(v_1, \dots, v_n)$ for the set (simplicial cone) in $\mathbb{R}^n$ defined by

$$\operatorname{span}^+(v_1, \dots, v_n) = \left\{ w \in \mathbb{R}^n : \exists \lambda_1, \dots, \lambda_n > 0 \text{ s.t. } w = \sum_{i=1}^{n} \lambda_i v_i \right\},$$

which is the natural restriction of $\operatorname{span}(v_1, \dots, v_n)$ to positive weights.

Lemma 29. *Let* $V_1, V_2, V_3, Y, Z \in \mathbb{R}^3$ *with* $\|Y\| = \|Z\|$ *and* $Y, Z \in \operatorname{span}^+(V_1, V_2, V_3)$. *Then at least one of the following inequalities does not hold:*

$$\begin{aligned} \langle V_1, Y\rangle &> \langle V_1, Z\rangle, \\ \langle V_2, Y\rangle &> \langle V_2, Z\rangle, \\ \langle V_3, Y\rangle &> \langle V_3, Z\rangle. \end{aligned}$$

Proof. See [SY25], Lemma 23. $\square$

Lemma 30. *For* $A, \overline{A}, B, \overline{B} \in \mathbb{R}^{n\times n}$ *and* $P_1, P_2 \in \mathbb{R}^n$ *it holds that*

$$|\langle AP_1, BP_2\rangle - \langle \overline{A}P_1, \overline{B}P_2\rangle| \le \|P_1\| \cdot \|P_2\| \cdot \Big(\|A - \overline{A}\| \cdot \|\overline{B}\| + \|\overline{A}\| \cdot \|B - \overline{B}\| + \|A - \overline{A}\| \cdot \|B - \overline{B}\| \Big).$$

Proof. See [SY25], Lemma 24. $\square$

Lemma 31. *For* $A, B \in \mathbb{R}^{n\times n}$ *and* $P_1, P_2 \in \mathbb{R}^n$ *one has*

$$|\langle AP_1, AP_2\rangle - \langle BP_1, BP_2\rangle| \le \|P_1\| \cdot \|P_2\| \cdot \|A - B\| \cdot \Big(\|A\| + \|B\| + \|A - B\| \Big).$$

Proof. See [SY25], Lemma 25. $\square$

Lemma 32. *Let* $A, B, C \in \mathbb{R}^2$ *be such that* $\langle R(\pi/2)A, B\rangle, \langle R(\pi/2)B, C\rangle, \langle R(\pi/2)C, A\rangle > 0$. *Then the origin lies strictly in the triangle* ABC.

Proof. See [SY25], Lemma 26. $\square$

Definition 33. Let $\theta, \varphi \in \mathbb{R}$, $\varepsilon > 0$, and set $M := M(\theta, \varphi)$. Three points $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ are called ε-spanning for (θ, φ) if it holds that:

$$\langle R(\pi/2)MP_1, MP_2\rangle > 2\varepsilon(\sqrt{2}+\varepsilon),$$
$$\langle R(\pi/2)MP_2, MP_3\rangle > 2\varepsilon(\sqrt{2}+\varepsilon),$$
$$\langle R(\pi/2)MP_3, MP_1\rangle > 2\varepsilon(\sqrt{2}+\varepsilon).$$

Lemma 34. *Let $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ be ε-spanning for $(\overline{\theta}, \overline{\varphi})$ and let $\theta, \varphi \in \mathbb{R}$ such that $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$. Assume that $\langle X(\theta, \varphi), P_i\rangle > 0$ for $i = 1, 2, 3$. Then*

$$X(\theta, \varphi) \in \operatorname{span}^+(P_1, P_2, P_3).$$

Proof. See [SY25], Lemma 28. □

Lemma 35. *Let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$. Let $\varepsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$, then set*

$$T := \left(R(\overline{\alpha})M(\overline{\theta}_1, \overline{\varphi}_1)P + M(\overline{\theta}_2, \overline{\varphi}_2)Q\right)/2 \in \mathbb{R}^2,$$

and $\delta \geq \|T - M(\overline{\theta}_2, \overline{\varphi}_2)Q\|$. Finally, let $\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha \in \mathbb{R}$ with $|\overline{\theta}_1 - \theta_1|, |\overline{\varphi}_1 - \varphi_1|, |\overline{\theta}_2 - \theta_2|, |\overline{\varphi}_2 - \varphi_2|, |\overline{\alpha} - \alpha| \leq \varepsilon$. Then $R(\alpha)M(\theta_1, \varphi_1)P, M(\theta_2, \varphi_2)Q \in \operatorname{Disc}_{\delta+\sqrt{5}\varepsilon}(T)$.

Proof. See [SY25], Lemma 30. □

Definition 36. Let $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon and $Q \in \mathcal{P}$ one of its vertices. Assume that for some $\overline{Q} \in \mathbb{R}^2$ it holds that $Q \in \operatorname{Disc}_\delta(\overline{Q})$, i.e. $\|Q - \overline{Q}\| < \delta$. Define $\operatorname{Sect}_\delta(\overline{Q}) := \operatorname{Disc}_\delta(\overline{Q}) \cap \mathcal{P}^\circ$ as the intersection between $\operatorname{Disc}_\delta(\overline{Q})$ and the interior of the convex hull of $\mathcal{P}$.

Moreover, $Q \in \mathcal{P}$ is called *δ-locally maximally distant with respect to $\overline{Q}$ (δ-LMD($\overline{Q}$))* if for all $A \in \operatorname{Sect}_\delta(\overline{Q})$ it holds that $\|Q\| > \|A\|$.

Lemma 37. *Let $\mathcal{P}$ be a convex polygon and $Q \in \mathcal{P}$ be one of its vertices. Let $\overline{Q} \in \mathbb{R}^2$ with $\|Q - \overline{Q}\| < \delta$ for some $\delta > 0$. Assume that for some $r > 0$ such that $\|Q\| > r$ it holds that*

$$\frac{\langle Q, Q - P_j\rangle}{\|Q\|\|Q - P_j\|} \geq \delta/r,$$

for all other vertices $P_j \in \mathcal{P} \setminus Q$. Then $Q \in \mathcal{P}$ is δ-locally maximally distant with respect to $\overline{Q}$.

Proof. See [SY25], Lemma 32. □

Lemma 38. *Let $\varepsilon > 0$ and $\theta, \overline{\theta}, \varphi, \overline{\varphi} \in \mathbb{R}$ with $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$. Define $M = M(\theta, \varphi)$ and $\overline{M} = M(\overline{\theta}, \overline{\varphi})$ and let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$. Assume that*

$$\frac{\langle \overline{M}P, \overline{M}(P-Q)\rangle - 2\varepsilon\|P-Q\| \cdot (\sqrt{2}+\varepsilon)}{(\|\overline{M}P\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M}(P-Q)\| + 2\sqrt{2}\varepsilon)} > 0.$$

Then:

$$\frac{\langle MP, M(P-Q)\rangle}{\|MP\| \cdot \|M(P-Q)\|} \geq \frac{\langle \overline{M}P, \overline{M}(P-Q)\rangle - 2\varepsilon\|P-Q\| \cdot (\sqrt{2}+\varepsilon)}{(\|\overline{M}P\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M}(P-Q)\| + 2\sqrt{2}\varepsilon)}.$$

Proof. See [SY25], Lemma 33. □

Lemma 39. *Let $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbb{R}^3$. Define the 3×3 matrices $P := (P_1|P_2|P_3)$ and $Q := (Q_1|Q_2|Q_3)$ and assume that Q is invertible. Then P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent if and only if $P^t P = Q^t Q$.*

Proof. From [SY25], Lemma 35. Note that $P^t P = Q^t Q$ is equivalent to saying that $\langle P_i, P_j \rangle = \langle Q_i, Q_j \rangle$ for all $1 \leq i, j \leq 3$. Moreover, the condition on invertibility of Q can be dropped, however then the proof becomes somewhat less straightforward.

If P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent then $\langle P_i, P_j \rangle = \langle LQ_i, LQ_j \rangle = \langle Q_i, Q_j \rangle$, thus $P^t P = Q^t Q$. For the other direction, we claim that $L := PQ^{-1}$ is orthonormal and satisfies that $P_i = LQ_i$ for all $i = 1, 2, 3$. Indeed, $L^t L = (PQ^{-1})^t (PQ^{-1}) = (Q^t)^{-1} P^t P Q^{-1} = \mathrm{Id}$ and it holds that $LQ = PQ^{-1}Q = P$, thus $LQ_i = P_i$. □

Theorem 40 (Local Theorem)**.** *Let $\mathbf{P}$ be a polyhedron with radius $\rho = 1$ and $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$ be not necessarily distinct. Assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent.*

Let $\varepsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$, then set $\overline{X_1} := X(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} := X(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that

$$(-1)^{\sigma_P} \langle \overline{X_1}, P_i \rangle > \sqrt{2}\varepsilon \quad \text{and} \quad (-1)^{\sigma_Q} \langle \overline{X_2}, Q_i \rangle > \sqrt{2}\varepsilon, \tag{A$_\varepsilon$}$$

for all $i = 1, 2, 3$. Moreover, assume that P_1, P_2, P_3 are ε-spanning for $(\overline{\theta}_1, \overline{\varphi}_1)$ and that Q_1, Q_2, Q_3 are ε-spanning for $(\overline{\theta}_2, \overline{\varphi}_2)$. Finally, assume that for all $i = 1, 2, 3$ and any $Q_j \in \mathbf{P} \setminus Q_i$ it holds that

$$\frac{\langle \overline{M_2} Q_i, \overline{M_2}(Q_i - Q_j) \rangle - 2\varepsilon \|Q_i - Q_j\| \cdot (\sqrt{2} + \varepsilon)}{(\|\overline{M_2} Q_i\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M_2}(Q_i - Q_j)\| + 2\sqrt{2}\varepsilon)} > \frac{\sqrt{5}\varepsilon + \delta}{r}, \tag{B$_\varepsilon$}$$

for some $r > 0$ such that $\min_{i=1,2,3} \|\overline{M_2} Q_i\| > r + \sqrt{2}\varepsilon$ and for some $\delta \in \mathbb{R}$ with

$$\delta \geq \max_{i=1,2,3} \left\| R(\overline{\alpha}) \overline{M_1} P_i - \overline{M_2} Q_i \right\| / 2.$$

Then there exists no solution to Rupert's problem $R(\alpha) M(\theta_1, \varphi_1) \mathbf{P} \subset M(\theta_2, \varphi_2) \mathbf{P}^\circ$ with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] := U \subseteq \mathbb{R}^5.$$

Proof. See [SY25], Theorem 36. □

Chapter 7

Rational Versions

Definition 41. We define the two functions $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ by:

$$\sin_{\mathbb{Q}}(x) := x - \frac{x^3}{3} + \frac{x^5}{5!} \mp \cdots + \frac{x^{25}}{25!},$$
$$\cos_{\mathbb{Q}}(x) := 1 - \frac{x^2}{2} + \frac{x^4}{4!} \mp \cdots + \frac{x^{24}}{24!}.$$

Further, by replacing $\sin, \cos$ with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ we define the functions

$$R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta, \varphi), M_{\mathbb{Q}}(\theta, \varphi), M^{\theta}_{\mathbb{Q}}(\theta, \varphi), M^{\varphi}_{\mathbb{Q}}(\theta, \varphi).$$

Lemma 42.

$$|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \frac{|x|^{27}}{27!} \quad \textit{and} \quad |\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \frac{|x|^{26}}{26!}.$$

Proof. Appeal to Taylor series bounds, using the fact that all absolute values of higher derivatives of sine and cosine never exceed 1. □

Lemma 43. *For every $x \in [-4, 4]$ it holds that*

$$|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \frac{\kappa}{7} \quad \textit{and} \quad |\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \frac{\kappa}{7}.$$

Proof. Straightforward numerical calculation from Lemma 42. □

Lemma 44. *Let $A = (a_{i,j})_{1\leq i\leq m,\ 1\leq j\leq n} \in \mathbb{R}^{m\times n}$ and $\delta > 0$. Assume that $|a_{i,j}| \leq \delta$. Then it holds that $\|A\| \leq \delta\sqrt{mn}$.*

Proof. For any $v \in \mathbb{R}^n$ we have

$$\|Av\|^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} v_j \right)^2 \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \delta |v_j| \right)^2 = \delta^2 m \left(\sum_{j=1}^{n} |v_j| \right)^2 \leq \delta^2 mn \|v\|^2$$

using the Cauchy-Schwarz inequality. Dividing by $\|v\|$ and taking the square root proves the claim. □

Lemma 45. *Let $A(x,y)$ be an $m \times n$ matrix with $1 \le m,n \le 3$ such that every entry is of the form $a_1(x) \cdot a_2(y)$ where $a_i(z) \in \{0,1,-1,\pm\sin(z),\pm\cos(z)\}$. Define $A_{\mathbb{Q}}(x,y)$ by replacing* $\sin$ *with* $\sin_{\mathbb{Q}}$ *and* $\cos$ *with* $\cos_{\mathbb{Q}}$. *Then for every $x,y \in [-4,4]$ it holds that $\|A(x,y) - A_{\mathbb{Q}}(x,y)\| \le \kappa$.*

Proof. We've replaced the assumption $a_i(z) \in \{0,1,-1,\pm\sin(z),\pm\cos(z)\}$ in [SY25]'s Lemma 40 with $a_i(z) \in [-1,1]$.

By assumption, for fixed x,y every entry of $A(x,y) - A_{\mathbb{Q}}(x,y)$ is of the form $ab - \tilde{a}\tilde{b}$ for some $a,b \in [-1,1]$ and $|a-\tilde{a}|, |b-\tilde{b}| \le \kappa/7$ by Theorem 43. This implies that

$$|ab - \tilde{a}\tilde{b}| \le |ab - a\tilde{b}| + |a\tilde{b} - \tilde{a}\tilde{b}| = |a| \cdot |b-\tilde{b}| + |\tilde{b}| \cdot |a-\tilde{a}| \le 1 \cdot \kappa/7 + (1+\kappa/7)\cdot\kappa/7 < \kappa/3.$$

So we can apply Theorem 44 and obtain that $\|A(x,y) - A_{\mathbb{Q}}(x,y)\| < \kappa/3 \cdot \sqrt{3 \cdot 3} = \kappa$. ☐

Corollary 46. *Let $\alpha,\theta,\varphi \in [-4,4]$. Then it holds that*

$$\|R(\alpha) - R_{\mathbb{Q}}(\alpha)\|, \|R'(\alpha) - R'_{\mathbb{Q}}(\alpha)\|, \|X(\theta,\varphi) - X_{\mathbb{Q}}(\theta,\varphi)\|, \|M(\theta,\varphi) - M_{\mathbb{Q}}(\theta,\varphi)\|, \\ \|M^{\theta}(\theta,\varphi) - M^{\theta}_{\mathbb{Q}}(\theta,\varphi)\|, \|M^{\varphi}(\theta,\varphi) - M^{\varphi}_{\mathbb{Q}}(\theta,\varphi)\| \le \kappa.$$

Moreover,

$$\|R_{\mathbb{Q}}(\alpha)\|, \|R'_{\mathbb{Q}}(\alpha)\|, \|M_{\mathbb{Q}}(\theta,\varphi)\|, \|M^{\theta}_{\mathbb{Q}}(\theta,\varphi)\|, \|M^{\varphi}_{\mathbb{Q}}(\theta,\varphi)\| \le 1+\kappa$$

Proof. The first statement is a direct application of Theorem 45 and the second statement follows immediately after using Theorem 9 and the triangle inequality. The derivative norm bounds follow similarly, using that the operator norms of R', M^{θ}, and M^{φ} are at most 1. ☐

Lemma 47. *For $1 \le i \le n$ let (A_i, B_i) be pairs of real matrices, such that for each i the dimensions of A_i and B_i are equal. Assume moreover that the products $A_1 \cdots A_n$ and $B_1 \cdots B_n$ are well defined. Finally, assume that $\|A_i - B_i\| \le \kappa$ and let $\delta_i \ge \max(\|A_i\|, \|B_i\|, 1)$. Then it holds that $\|A_1 \cdots A_n - B_1 \cdots B_n\| \le n\kappa \cdot \delta_1 \cdots \delta_n$.*

Proof. See [SY25], Lemma 42. ☐

Lemma 48. *Let $\alpha,\theta,\varphi \in [-4,4]$, $P \in \mathbb{R}^3$ with $\|P\| \le 1$ and let $\widetilde{P}$ be a κ-rational approximation of P. Set $M = M(\theta,\varphi)$ and $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta,\varphi)$, $M^{\theta} = M^{\theta}(\theta,\varphi)$, $M^{\theta}_{\mathbb{Q}} = M^{\theta}_{\mathbb{Q}}(\theta,\varphi)$, $M^{\varphi} = M^{\varphi}(\theta,\varphi)$, $M^{\varphi}_{\mathbb{Q}} = M^{\varphi}_{\mathbb{Q}}(\theta,\varphi)$ as well as $R = R(\alpha)$, $R_{\mathbb{Q}} = R_{\mathbb{Q}}(\alpha)$, $R' = R'(\alpha)$, $R'_{\mathbb{Q}} = R'_{\mathbb{Q}}(\alpha)$. Finally let $w \in \mathbb{R}^2$ with $\|w\| = 1$. Then:*

$$|\langle MP, w\rangle - \langle M_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 3\kappa, \tag{7.1}$$

$$|\langle M^{\theta}P, w\rangle - \langle M^{\theta}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 3\kappa, \tag{7.2}$$

$$|\langle M^{\varphi}P, w\rangle - \langle M^{\varphi}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 3\kappa, \tag{7.3}$$

$$|\langle RMP, w\rangle - \langle R_{\mathbb{Q}}M_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa, \tag{7.4}$$

$$|\langle R'MP, w\rangle - \langle R'_{\mathbb{Q}}M_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa, \tag{7.5}$$

$$|\langle RM^{\theta}P, w\rangle - \langle R_{\mathbb{Q}}M^{\theta}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa, \tag{7.6}$$

$$|\langle RM^{\varphi}P, w\rangle - \langle R_{\mathbb{Q}}M^{\varphi}_{\mathbb{Q}}\widetilde{P}, w\rangle| \le 4\kappa. \tag{7.7}$$

Proof. See [SY25], Lemma 44. ☐

Theorem 49 (Rational Global Theorem)**.** *Let* $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* $\rho = 1$ *and* $\widetilde{\mathbf{P}}$ *a* κ*-rational approximation. Let* $\widetilde{S} \in \widetilde{\mathbf{P}}$*. Further let* $\varepsilon > 0$ *and* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$*. Let* $w \in \mathbb{Q}^2$ *be a unit vector. Denote* $\overline{M_1} := M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_2} := M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^{\theta} := M_{\mathbb{Q}}^{\theta}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_1}^{\varphi} := M_{\mathbb{Q}}^{\varphi}(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^{\theta}, \overline{M_2}^{\varphi}$*. Finally set*

$$G^{\mathbb{Q}} := \langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle - \varepsilon \cdot \left(|\langle R_{\mathbb{Q}}'(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^{\theta}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^{\varphi}\widetilde{S}, w\rangle|\right) - 9\varepsilon^2/2 - 4\kappa(1 + 3\varepsilon),$$

$$H_P^{\mathbb{Q}} := \langle \overline{M_2}P, w\rangle + \varepsilon \cdot \left(|\langle \overline{M_2}^{\theta}P, w\rangle| + |\langle \overline{M_2}^{\varphi}P, w\rangle|\right) + 2\varepsilon^2 + 3\kappa(1 + 2\varepsilon).$$

If $G^{\mathbb{Q}} > \max_{P \in \widetilde{\mathbf{P}}} H_P^{\mathbb{Q}}$ *then there does not exist a solution to Rupert's condition to* $\mathbf{P}$ *with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon].$$

Proof. □

Definition 50. Let $\theta, \varphi \in \mathbb{Q} \cap [-4, 4]$ and $M_{\mathbb{Q}} := M_{\mathbb{Q}}(\theta, \varphi)$. Three points $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3 \in \mathbb{Q}^3$ with $\|\widetilde{P}_1\|, \|\widetilde{P}_2\|, \|\widetilde{P}_3\| \le 1 + \kappa$ are called *ε-κ-spanning for* (θ, φ) if it holds that:

$$\langle R(\pi/2)M_{\mathbb{Q}}\widetilde{P}_1, M_{\mathbb{Q}}\widetilde{P}_2\rangle > 2\varepsilon(\sqrt{2} + \varepsilon) + 6\kappa,$$
$$\langle R(\pi/2)M_{\mathbb{Q}}\widetilde{P}_2, M_{\mathbb{Q}}\widetilde{P}_3\rangle > 2\varepsilon(\sqrt{2} + \varepsilon) + 6\kappa,$$
$$\langle R(\pi/2)M_{\mathbb{Q}}\widetilde{P}_3, M_{\mathbb{Q}}\widetilde{P}_1\rangle > 2\varepsilon(\sqrt{2} + \varepsilon) + 6\kappa.$$

Lemma 51. *Let* $P_1, P_2, P_3 \in \mathbb{R}^3$ *with* $\|P_i\| \le 1$ *and* $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3 \in \mathbb{Q}^3$ *be their* κ*-rational approximations. Assume that* $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ *are* ε*-*κ*-spanning for some* $\theta, \varphi \in \mathbb{Q} \cap [-4, 4]$*, then* P_1, P_2, P_3 *are* ε*-spanning for* θ, φ*.*

Proof. See [SY25], Lemma 46. □

Lemma 52. *Let* $P, Q \in \mathbb{R}^3$ *with* $\|P\|, \|Q\| \le 1$ *and* $\widetilde{P}, \widetilde{Q}$ *some respective* κ*-rational approximations. Moreover, let* $\alpha, \theta, \varphi \in \mathbb{R} \in [-4, 4]$ *and set* $X = X(\theta, \varphi)$*,* $X_{\mathbb{Q}} = X_{\mathbb{Q}}(\theta, \varphi)$ *as well as* $M = M(\theta, \varphi)$*,* $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta, \varphi)$*. Then*

$$|\langle X, P\rangle - \langle X_{\mathbb{Q}}, \widetilde{P}\rangle| \le 3\kappa, \tag{7.8}$$

$$|\langle MP, MQ\rangle - \langle M_{\mathbb{Q}}\widetilde{P}, M_{\mathbb{Q}}\widetilde{Q}\rangle| \le 5\kappa, \tag{7.9}$$

$$|\|MQ\| - \|M_{\mathbb{Q}}\widetilde{Q}\|| \le 3\kappa. \tag{7.10}$$

Proof. See [SY25], Lemma 49. □

Corollary 53. *Let* $P, Q \in \mathbb{R}^3$ *with* $\|P\|, \|Q\| \le 1$ *and* $\widetilde{Q}$ *a* κ*-rational approximation of* Q*. Let* $\alpha, \theta, \varphi, \overline{\theta}, \overline{\varphi} \in [-4, 4]$ *and set* $M = M(\theta, \varphi)$*,* $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta, \varphi)$*,* $\overline{M} = M(\overline{\theta}, \overline{\varphi})$*,* $\overline{M}_{\mathbb{Q}} = M_{\mathbb{Q}}(\overline{\theta}, \overline{\varphi})$*. Then*

$$|\|R(\alpha)MP - \overline{M}Q\| - \|R_{\mathbb{Q}}(\alpha)M_{\mathbb{Q}}P - \overline{M}_{\mathbb{Q}}\widetilde{Q}\|| \le 6\kappa$$

Note that the rational side uses P *directly (not a rational approximation* $\widetilde{P}$*).*

Proof. See [SY25], Corollary 50. □

Corollary 54. *In the setting of Theorem 52, let* $\sqrt[+]{x}$ *be an upper square-root function, i.e.,* $\sqrt{x} \leq \sqrt[+]{x}$ *for all real* $x \geq 0$ *with rational output on rational input. Set* $\|x\|_+ := \sqrt[+]{\|x\|^2}$*. Set*

$$A = \frac{\langle MP, M(P-Q)\rangle - 2\varepsilon\|P-Q\| \cdot (\sqrt{2}+\varepsilon)}{(\|MP\| + \sqrt{2}\varepsilon) \cdot (\|M(P-Q)\| + 2\sqrt{2}\varepsilon)}$$

as well as

$$A_{\mathbb{Q}} = \frac{\langle M_{\mathbb{Q}}\widetilde{P}, M_{\mathbb{Q}}(\widetilde{P}-\widetilde{Q})\rangle - 10\kappa - 2\varepsilon(\|\widetilde{P}-\widetilde{Q}\| + 2\kappa) \cdot (\sqrt{2}+\varepsilon)}{(\|M_{\mathbb{Q}}\widetilde{P}\|_+ + \sqrt{2}\varepsilon + 3\kappa) \cdot (\|M_{\mathbb{Q}}(\widetilde{P}-\widetilde{Q})\|_+ + 2\sqrt{2}\varepsilon + 6\kappa)}.$$

Assume that $A \geq 0$*. Then it holds that* $A \geq A_{\mathbb{Q}}$*.*

Proof. See [SY25], Corollary 51. □

Theorem 55 (Rational Local Theorem)**.** *Let* $\mathbf{P}$ *be a polyhedron with radius* $\rho = 1$ *and* $\widetilde{P}_i$ *be a* κ*-rational approximation of* $P_i \in \mathbf{P}$*. Set* $\widetilde{\mathbf{P}} = \{\widetilde{P}_i \text{ for } P_i \in \mathbf{P}\}$*. Let* $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$ *be not necessarily distinct and assume that* P_1, P_2, P_3 *and* Q_1, Q_2, Q_3 *are congruent. Let* $\varepsilon > 0$ *and* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$*. Set* $\overline{X_1} := X_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} := X_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1} := M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} := M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$*. Assume that there exist* $\sigma_P, \sigma_Q \in \{0, 1\}$ *such that*

$$(-1)^{\sigma_P}\langle \overline{X_1}, \widetilde{P}_i\rangle > \sqrt{2}\varepsilon + 3\kappa \quad \textit{and} \quad (-1)^{\sigma_Q}\langle \overline{X_2}, \widetilde{Q}_i\rangle > \sqrt{2}\varepsilon + 3\kappa, \qquad (\mathrm{A}_\varepsilon^{\mathbb{Q}})$$

for all $i = 1, 2, 3$*. Moreover, assume that* $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ *are* ε*-*κ*-spanning for* $(\overline{\theta}_1, \overline{\varphi}_1)$ *and that* $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3$ *are* ε*-*κ*-spanning for* $(\overline{\theta}_2, \overline{\varphi}_2)$*. Let* $\sqrt[+]{x}$ *and* $\sqrt[-]{x}$ *be upper- and lower-square-root functions (bounding* $\sqrt{x}$ *from above/below for all real* $x \geq 0$*, with rational output on rational input), then set* $\|Z\|_+ := \sqrt[+]{\|Z\|^2}$ *and* $\|Z\|_- := \sqrt[-]{\|Z\|^2}$ *for* $Z \in \mathbb{R}^n$*. Finally, assume that for all* $i = 1, 2, 3$ *and any* $\widetilde{Q}_j \in \widetilde{\mathbf{P}} \setminus \widetilde{Q}_i$ *it holds that*

$$\frac{\langle \overline{M_2}\widetilde{Q}_i, \overline{M_2}(\widetilde{Q}_i - \widetilde{Q}_j)\rangle - 10\kappa - 2\varepsilon(\|\widetilde{Q}_i - \widetilde{Q}_j\|_+ + 2\kappa) \cdot (\sqrt{2}+\varepsilon)}{(\|\overline{M_2}\widetilde{Q}_i\|_+ + \sqrt{2}\varepsilon + 3\kappa) \cdot (\|\overline{M_2}(\widetilde{Q}_i - \widetilde{Q}_j)\|_+ + 2\sqrt{2}\varepsilon + 6\kappa)} > \frac{\sqrt{5}\varepsilon + \delta}{r}, \qquad (\mathrm{B}_\varepsilon^{\mathbb{Q}})$$

for some $r > 0$ *such that* $\min_{i=1,2,3} \|\overline{M_2}\widetilde{Q}_i\|_- > r + \sqrt{2}\varepsilon + 3\kappa$ *and for some* $\delta \in \mathbb{R}$ *with*

$$\delta \geq \max_{i=1,2,3} \left\| R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{P}_i - \overline{M_2}\widetilde{Q}_i \right\|_+ / 2 + 3\kappa.$$

Then there exists no solution to Rupert's problem $R(\alpha)M(\theta_1, \varphi_1)\,\mathbf{P} \subset M(\theta_2, \varphi_2)\,\mathbf{P}^\circ$ *with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5.$$

Proof. , □

Chapter 8

Computational Step

Definition 56. We define rational approximations of the 90 noperthedron vertices by $\lfloor x \cdot 10^{16} \rfloor / 10^{16}$.

Theorem 57. `nopertQ is a` κ`-rational approximation of the Noperthedron.`

Proof. □

Theorem 58. *There exists a valid solution table whose zeroth row covers*

$$\begin{aligned} \theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58]. \end{aligned}$$

Proof. By exhibiting the table and running the validity checking algorithm. □

Theorem 59. *If a global node in the solution tree is valid, then there is no Rupert solution for its interval.*

Proof. □

Theorem 60. *If a local node in the solution tree is valid, then there is no Rupert solution for its interval.*

Proof. □

Theorem 61. *If we have a valid solution table, and in particular its ith row is valid, then there is no Rupert solution of the interval of its ith row.*

Proof. By a mutual induction on the number of rows left in the table following the ith. This is because validity constrains each row to only refer to later entries. Appeal inductively to this same theorem if the row splits into other nodes in the tree, or appeal to Theorem 59 or Theorem 60) at the leaves. □

Corollary 62. *If we have a valid solution table, then there is no Rupert solution of the interval of its zeroth row.*

Proof. Immediate special case of Theorem 61. □

Chapter 9

Main Theorems

Theorem 63. *There does not in fact exist a noperthedron Rupert solution with*

$$\begin{aligned} \theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58]. \end{aligned}$$

Proof. By 58, there is a valid solution table containing a valid row whose pose interval is a superset of the 5-d interval above. By 62, this means there is no Rupert solution in that interval. □

Theorem 64. *There is no 5-parameter pose that makes the noperthedron have the Rupert property.*

Proof. Theorem 63 says there is no tight pose that makes the noperthedron Rupert. Corollary 8 says that this suffices for the general case. □

Theorem 65. *There is no purely rotational pose that makes the noperthedron have the Rupert property.*

Proof. Suppose there were a purely rotational pose. Then convert that to an equivalent 5-parameter pose with Theorem 19 and appeal to Theorem 64. □

Theorem 66. *There is no pose that makes the noperthedron have the Rupert property.*

Proof. By Theorem 20, we need only show that the noperthedron is pointsymmetric to see that if it is Rupert, then it must be Rupert via a purely rotational pose. But Lemma 6 shows exactly this. And yet we know via Theorem 65 that the noperthedron is not rotationally Rupert, so we have a contradiction, hence the noperthedron has no pose that makes it Rupert. □

Theorem 67. *The noperthedron is not a Rupert set.*

Proof. By Theorem 66, there is no pose that makes the noperthedron a Rupert set. □

Theorem 68. *The noperthedron is not a Rupert polyhedron.*

Proof. By Theorem 18 it suffices to show that the convex hull of the noperthedron vertices is not a Rupert set. But this is exactly what Theorem 67 shows. □

Bibliography

[SY25] Jakob Steininger and Sergey Yurkevich. A convex polyhedron without rupert's property. https://arxiv.org/abs/2508.18475, 2025.